Math 432 lec 13 Trees and Cayley's Formula

(1) Definition: a *tree* is a connected graph without any cycle.

(2) Properties of trees: every tree with at least 2 vertices has at least 2 leaves. (pf: take a longest path in the tree. Then the endpoints have no neighbors outside of the path (otherwise it is not the longest) and no neighbors on the path (otherwise there is a cycle). So the endpoints have degree 1, thus are leaves.)

(3) The following are equivalent: Let G be a vertex with n vertices
(a) G is a tree;
(b) G is connected and has exactly $n-1$ edges.
(c) G is connected and every edge of the graph is a bridge.
(d) every pair of distinct vertices x and y in G is joined by a unique path.

Pf: Let G be a tree. Then G is connected, and there is a path between any pair of distinct vertices. To show (b), we use induction on the number of vertices. We just showed that G has a leaf, say u. Then $G-u$ is still a tree. By induction, $G-u$ has $n-2$ edges, so G has $n-1$ edges. To show (c), we observe that if $e=uv$ is not a bridge, then there is an u,v-path P in $G-e$. But $P+e$ is a cycle in G, a contradiction. To show (c), if there are two paths between x and y, then the two paths together form a closed walk, so it must contain a cycle, a contradiction.

Now assume the truth of (b). We use induction on n again. As $e(G)=n-1$, $2e(G)=\sum_u d(u)=2n-2<2n$, so some vertex, say x, has degree 1. Now $G-x$ is connected and has $n-2$ edges, so $G-x$ is a tree by induction. Then G has no cycle as well. So G is a tree.

Assume (c) now. Clearly G cannot contain a cycle, for otherwise, the edges on the cycle are not bridges. So G is a tree.

Assume (d) now. Clearly G cannot contain a cycle, for otherwise, there are two paths between vertices on the cycle. So G is a tree.

(4) Spanning tree in a connected graph:
(a) a spanning tree in a graph is a tree contained in the graph which contains all the vertices.
(b) algorithms to find a spanning tree:
- remove the non-bridges one-by-one; or
- breath-first search; or
- add edges one by one so that no cycle is formed.

(c) find minimal spanning in a weighted graph: Kruskal's algorithm.

(5) a bipartite graph G is a graph whose vertices can be partitioned into two parts (A and B) so that all the edges are between the two parts.

A tree is a bipartite graphs. (pf: start from a vertex, do a BFS search, and label the vertices in each level alternatively by 0 and 1.)